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Chapter 3: Linear Equations

Chapter 4: Introduction to Functions

Foreword

Math Mammoth Grade 8 comprises a complete math curriculum for the eighth grade mathematics studies. The curriculum meets the Common Core standards.

In 8th grade, students spend the majority of the time with algebraic topics, such as linear equations, functions, and systems of equations. The other major topics are geometry and statistics.

The main areas of study in Math Mammoth Grade 8 are:

- Exponents laws and scientific notation
- Square roots, cube roots, and irrational numbers
- Geometry: congruent transformations, dilations, angle relationships, volume of certain solids, and the Pythagorean Theorem
- Solving and graphing linear equations;
- Introduction to functions;
- Systems of linear equations;
- Scatter plots/bivariate data.

We start with a study of exponent laws, using both numerical and algebraic expressions. The first chapter also covers scientific notation (both with large and small numbers), significant digits, and calculations with numbers given in scientific notations.

In chapter 2, students learn about geometric transformations (translations, reflections, rotations, dilations), common angle relationships, and volume of prisms, cylinders, spheres, and cones.

Next, in chapter 3, our focus is on linear equations. Students both review and learn more about solving linear equations, including equations whose solutions require the usage of the distributive property and equations where the variable is on both sides.

Chapter 4 presents an introduction to functions. Students construct functions to model linear relationships, learn to use the rate of change and initial value of the function, and they describe functions qualitatively based on their graphs.

In part 8-B, students graph linear equations, learn about irrational numbers and the Pythagorean Theorem, solve systems of linear equations, and investigate patterns of association in bivariate data (scatter plots).

I heartily recommend that you read the full user guide in the following pages.

I wish you success in teaching math!

Maria Miller, the author

User Guide

Note: You can also find the information that follows online, at **https://www.mathmammoth.com/userguides/** .

Basic principles in using Math Mammoth Complete Curriculum

Math Mammoth is mastery-based, which means it concentrates on a few major topics at a time, in order to study them in depth. The two books (parts A and B) are like a “framework”, but you still have some liberty in planning your student’s studies. In eighth grade, chapters 2 (geometry), 3 (linear equations) and chapter 4 (functions) should be studied before chapter 5 (graphing linear equations). Also, chapters 3, 4, and 5 should be studied before chapter 7 (systems of linear equations) and before chapter 8 (statistics). However, you still have some flexibility in scheduling the various chapters.

Math Mammoth is not a scripted curriculum. In other words, it is not spelling out in exact detail what the teacher is to do or say. Instead, Math Mammoth gives you, the teacher, various tools for teaching:

- **The two student worktexts** (parts A and B) contain all the lesson material and exercises. They include the explanations of the concepts (the teaching part) in blue boxes. The worktexts also contain some advice for the teacher in the “Introduction” of each chapter.

  The teacher can read the teaching part of each lesson before the lesson, or read and study it together with the student in the lesson, or let the student read and study on his own. If you are a classroom teacher, you can copy the examples from the “blue teaching boxes” to the board and go through them on the board.

- Don’t automatically assign all the exercises. Use your judgement, trying to assign just enough for your student’s needs. You can use the skipped exercises later for review. For most students, I recommend to start out by assigning about half of the available exercises. Adjust as necessary.

- For each chapter, there is a **link list to various free online games** and activities. These games can be used to supplement the math lessons, for learning math facts, or just for some fun. Each chapter introduction (in the student worktext) contains a link to the list corresponding to that chapter.

- The student books contain some **mixed review lessons**, and the curriculum also provides you with additional **cumulative review lessons**.

- There is a **chapter test** for each chapter of the curriculum, and a comprehensive end-of-year test.

- You can use the free online exercises at https://www.mathmammoth.com/practice/
  This is an expanding section of the site, so check often to see what new topics we are adding to it!

- And there are answer keys to everything.

How to get started

Have ready the first lesson from the student worktext. Go over the first teaching part (within the blue boxes) together with your student. Go through a few of the first exercises together, and then assign some problems for the student to do on their own.

Repeat this if the lesson has other blue teaching boxes.

Many students can eventually study the lessons completely on their own — the curriculum becomes self-teaching. However, students definitely vary in how much they need someone to be there to actually teach them.

Pacing the curriculum

Worktext 8-A	
Chapter 1	13 days
Chapter 2	27 days
Chapter 3	21 days
Chapter 4	14 days
TOTAL	**75 days**

Each chapter introduction contains a suggested pacing guide for that chapter. You will see a summary on the right. (This summary does not include time for optional tests.)

Most lessons are 3 or 4 pages long, intended for one day. Some lessons are 5 pages and can be covered in two days.

It can also be helpful to calculate a general guideline as to how many pages per week the student should cover in order to go through the curriculum in one school year.

The table below lists how many pages there are for the student to finish in this particular grade level, and gives you a guideline for how many pages per day to finish, assuming a 180-day (36-week) school year. The page count in the table below *includes* the optional lessons.

Example:

Grade level	School days	Days for tests and reviews	Lesson pages	Days for the student book	Pages to study per day	Pages to study per week
8-A	90	8	204	82	2.5	12.5
8-B	90	8	182	82	2.2	11
Grade 8 total	180	16	386	164	2.4	12

The table below is for you to fill in. Allow several days for tests and additional review before tests — I suggest at least twice the number of chapters in the curriculum. Then, to get a count of "pages to study per day", **divide the number of lesson pages by the number of days for the student book**. Lastly, multiply this number by 5 to get the approximate page count to cover in a week.

Grade level	Number of school days	Days for tests and reviews	Lesson pages	Days for the student book	Pages to study per day	Pages to study per week
8-A			204			
8-B			182			
Grade 8 total			386			

Now, something important. Whenever the curriculum has lots of similar practice problems (a large set of problems), feel free to **only assign 1/2 or 2/3 of those problems**. If your student gets it with less amount of exercises, then that is perfect! If not, you can always assign the rest of the problems for some other day. In fact, you could even use these unassigned problems the next week or next month for some additional review.

In general, 8th graders might spend 45-75 minutes a day on math. If your student finds math enjoyable, they can of course spend more time with it! However, it is not good to drag out the lessons on a regular basis, because that can then affect the student's attitude towards math.

Using tests

For each chapter, there is a **chapter test**, which can be administered right after studying the chapter. **The tests are optional.** The main reason for the tests is for diagnostic purposes, and for record keeping. These tests are not aligned or matched to any standards.

In the digital version of the curriculum, the tests are provided both as PDF files and as html files. Normally, you would use the PDF files. The html files are included so you can edit them (in a word processor such as Word or LibreOffice), in case you want your student to take the test a second time. Remember to save the edited file under a different file name, or you will lose the original.

The end-of-year test is best administered as a diagnostic or assessment test, which will tell you how well the student remembers and has mastered the mathematics content of the entire grade level.

Using cumulative reviews and the worksheet maker

The student books contain mixed review lessons which review concepts from earlier chapters. The curriculum also comes with additional cumulative review lessons, which are just like the mixed review lessons in the student books, with a mix of problems covering various topics. These are found in their own folder in the digital version, and in the Tests & Cumulative Reviews book in the print version.

The cumulative reviews are optional; use them as needed. They are named indicating which chapters of the main curriculum the problems in the review come from. For example, “Cumulative Review, Chapter 4” includes problems that cover topics from chapters 1-4.

Both the mixed and cumulative reviews allow you to spot areas that the student has not grasped well or has forgotten. When you find such a topic or concept, you have several options:

1. Check for any online games and resources in the Introduction part of the particular chapter in which this topic or concept was taught.
2. If you have the digital version, you could reprint the lesson from the student worktext, and have the student restudy that.
3. Perhaps you only assigned 1/2 or 2/3 of the exercise sets in the student book at first, and can now use the remaining exercises.
4. Check if our online practice area at https://www.mathmammoth.com/practice/ has something for that topic.
5. Khan Academy has free online exercises, articles, and videos for most any math topic imaginable.

Concerning challenging word problems and puzzles

While this is not absolutely necessary, I heartily recommend supplementing Math Mammoth with challenging word problems and puzzles. You could do that once a month, for example, or more often if the student enjoys it.

The goal of challenging story problems and puzzles is to **develop the student’s logical and abstract thinking and mental discipline**. I recommend starting these in fourth grade, at the latest. Then, students are able to read the problems on their own and have developed mathematical knowledge in many different areas. Of course I am not discouraging students from doing such in earlier grades, either.

Math Mammoth curriculum contains lots of word problems, and they are usually multi-step problems. Several of the lessons utilize a bar model for solving problems. Even so, the problems I have created are usually tied to a specific concept or concepts. I feel students can benefit from solving problems and puzzles that require them to think “out of the box” or are just different from the ones I have written.

I recommend you use the free Math Stars problem-solving newsletters as one of the main resources for puzzles and challenging problems:

Math Stars Problem Solving Newsletter (grades 1-8)
https://www.homeschoolmath.net/teaching/math-stars.php

I have also compiled a list of other resources for problem solving practice, which you can access at this link:

https://l.mathmammoth.com/challengingproblems

Another idea: you can find puzzles online by searching for “brain puzzles for kids,” “logic puzzles for kids” or “brain teasers for kids.”

Frequently asked questions and contacting us

If you have more questions, please first check the FAQ at https://www.mathmammoth.com/faq-lightblue

If the FAQ does not cover your question, you can then contact us using the contact form at the Math Mammoth.com website.

Chapter 1: Exponents and Scientific Notation
Introduction

The first chapter of Math Mammoth Grade 8 starts out with a study of basic exponent laws and scientific notation.

We begin with a review of the concept of an exponent and of the order of operations. The next lesson first review multiplication of integers, and then focuses on powers with negative bases, such as $(-5)^3$.

Then we get to the "meat" of the chapter: the various laws of exponents. The first lesson on that topic allows students to explore and to find for themselves the product law and the quotient law of exponents. After that, students find out the logical way to define negative and zero exponent by looking at patterns. They practise simplifying various expressions with exponents, both with numerical values and with variables.

The lesson "More on Negative Exponents" focuses on expressions with a negative exponent in the numerator, such as $7/(a^{-4})$. This is to prepare students for calculations that ask them to find how many times bigger one number is than another, when the numbers are written in scientific notation.

Next, in the lesson "Laws of Exponents, Part 2", students practise applying the power of a power law: $(a^n)^m = a^{nm}$.

Then the chapter has one more lesson on the laws of exponents ("Laws of Exponents, Part 3"), which summarizes the laws and gives more practice. This lesson is not absolutely essential if you're following Common Core Standards. It is presented here to give a summary, to give practice on all exponent laws, including the power of a quotient law which was not dealt with a lot in the previous lessons. This lesson also allows the book to meet the Florida B.E.S.T. standards for 8th grade.

Then we turn our attention to scientific notation, first learning how it is used with large numbers and then with small numbers. The lesson on significant digits follows, helping students to know how to round final answers in calculations with measurements.

The last topic of the chapter is calculations with numbers given in scientific notations. These calculations, naturally, involve many scientific topics such as the atomic world or astronomy.

Pacing Suggestion for Chapter 1

This table does not include the chapter test as it is found in a different book (or file).
Please add one day to the pacing if you use the test.

The Lessons in Chapter 1	page	span	suggested pacing	your pacing
Scientific Notation: Large Numbers	32	*3 pages*	1 day	
Scientific Notation: Small Numbers	35	*2 pages*	1 day	
Significant Digits	37	*3 pages*	1 day	
Using Scientific Notation in Calculations, Part 1	40	*3 pages*	1 day	
Using Scientific Notation in Calculations, Part 2	43	*3 pages*	1 day	
Chapter 1 Review	46	*2 pages*	1 day	
Chapter 1 Test (optional)				
TOTALS		*35 pages*	13 days	

Helpful Resources on the Internet

We have compiled a list of Internet resources that match the topics in this chapter, including pages that offer:

- **online practice** for concepts;
- online **games**, or occasionally, printable games;
- **animations** and interactive **illustrations** of math concepts;
- **articles** that teach a math concept.

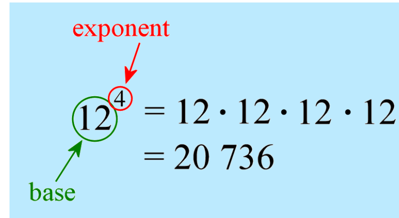
We heartily recommend you take a look! Many of our customers love using these resources to supplement the bookwork. You can use these resources as you see fit for extra practice, to illustrate a concept better and even just for some fun. Enjoy!

https://l.mathmammoth.com/gr8ch1

Scan me

Powers and the Order of Operations

You will recall that we use **exponents** as a shorthand for writing repeated multiplications by the same number. For example, $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$ is written 7^5.



The tiny raised number is called the **exponent**. It tells us how many times the **base** number is multiplied by itself.

The entire expression, 7^5, is a **power.** We read it as “seven to the fifth power,” “seven to the fifth,” or “seven raised to the fifth power.” Similarly, 0.5^8 is read as “five tenths to the eighth power” or “zero point five to the eighth.”

The “powers of 8” are the various expressions where 8 is raised to some power: for example, 8^3, 8^4, 8^{45}, and 8^{99} are powers of 8.

The expression 9^1 equals simply 9. In general, $a^1 = a$.

Powers of 2 are usually read as something “**squared.**” For example, 11^2 is read as “eleven squared.” That is because it gives us the area of a square with sides 11 units long.

Similarly, if the exponent is 3, the expression is usually read using the word “**cubed.**” For example, 1.5^3 is read as “one point five cubed” because it is the volume of a cube with edges 1.5 units long.

A calculator is not needed for the exercises of this lesson.

1. Evaluate.

a. four cubed **b.** 2^4 **c.** 5^3

d. 0.2^3 **e.** 1^{60} **f.** 100 squared

2. **a.** Which is more, 4^2 or 2^4? **b.** Which is more, 2^5 or 5^2?

3. Complete the patterns.

a.	b.	c.
$10^1 =$	$2^1 =$	$0.1^1 =$
$10^2 =$	$2^2 =$	$0.1^2 =$
$10^3 =$	$2^3 =$	$0.1^3 =$
$10^4 =$	$2^4 =$	$0.1^4 =$
$10^5 =$	$2^5 =$	$0.1^5 =$
$10^6 =$	$2^6 =$	$0.1^6 =$
$10^7 =$	$2^7 =$	$0.1^7 =$

The order of operations dictates that powers (expressions with exponents) are solved before multiplication, division, addition, and subtraction.

Example 1. Find the value of $5 \cdot 0.1^3 + 0.2^2$.

First the powers: $0.1^3 = 0.1 \cdot 0.1 \cdot 0.1 = 0.001$, and $0.2^2 = 0.2 \cdot 0.2 = 0.04$.

The expression becomes
$5 \cdot 0.001 + 0.04 \;=\; 0.005 + 0.04 \;=\; \underline{0.045}$.

The Order of Operations (BEMDAS)
("*Best Excuse My Dear Aunt Sally*")

1) Solve what is within brackets **(B)**.

2) Solve exponents **(E)**.

3) Solve multiplication **(M)** and division **(D)** from left to right.

4) Solve addition **(A)** and subtraction **(S)** from left to right.

4. Find the value of the expressions.

a. $4 \cdot 10^3 - 5 \cdot 10^2$	**b.** $4(5^2 - 2^3)$	**c.** $\dfrac{3}{1^8} + \dfrac{5}{3^2}$
d. $7 \cdot 10^3 - 5(800 - 10^2)$	**e.** $500 - \dfrac{3 \cdot 8}{2^3} + 2 \cdot 8^2$	**f.** $\dfrac{2 \cdot 17 + 2^4}{7 \cdot 7 - 3^2} + 20$

5. Find the value of the expressions.

a. $0.5^2 - 0.2^2 - 0.1^2$	**b.** $3(0.1^2 - 0.2^3)$	**c.** $0.6^2 + 2(1 - 0.3^2)$

6. The table on the right shows a list of powers of 4.

$4^1 = 4$
$4^2 = 16$
$4^3 = 64$
$4^4 = 256$
$4^5 = 1024$
$4^6 = 4096$

a. Find the value of 4^7 using the value for 4^6. (Do not use a calculator.)

b. Which power of 4 is equal to 65 536? Use estimation and the table, not a calculator.

c. Use the table to check whether $4^2 + 4^3 = 4^5$.

d. Use the table to check whether $4^2 \cdot 4^3 = 4^5$.

7. **a.** Find a power of 3 that is greater than seven squared.

b. Find a power of 5 that is greater than ten cubed.

c. Find a power of 1 that is greater than three squared.

8. **a.** If $3^6 = 729$, find the value of 3^8.

b. If $2^8 = 256$, find the value of 2^{11}.

9. Find the missing exponents.

a. $10^4 = 100^{\square}$	**b.** $2^6 = 4^{\square}$	**c.** $9^2 = 3^{\square}$	**d.** $0 = 0^{\square}$
e. $0.1^{\square} = 0.0001$	**f.** $0.2^{\square} = 0.00032$	**g.** $625 = 5^{\square}$	**h.** $128 = 2^{\square}$

10. Find the value of these powers.

a. $\left(\frac{1}{6}\right)^2 =$	**b.** $\left(\frac{3}{10}\right)^3 =$	**c.** $\left(\frac{2}{3}\right)^4 =$	**d.** $\left(\frac{3}{4}\right)^3 =$

Example 2. Simplify $3 \cdot s \cdot s \cdot s \cdot 3 \cdot t \cdot s \cdot t \cdot t$.

We can multiply in any order, so let's reorganise the expression as $3 \cdot 3 \cdot s \cdot s \cdot s \cdot s \cdot t \cdot t \cdot t$.

The variable s is multiplied by itself four times, and t three times. Naturally, $3 \cdot 3$ is 9.

So, we get $3 \cdot 3 \cdot s \cdot s \cdot s \cdot s \cdot t \cdot t \cdot t = \mathbf{9s^4t^3}$.

11. Simplify.

a. $2 \cdot x \cdot x \cdot x \cdot x \cdot x \cdot 7$	**b.** $4 \cdot x \cdot x \cdot x \cdot y \cdot y \cdot 9 \cdot x \cdot y \cdot x$
c. $5 \cdot a \cdot b \cdot b \cdot a \cdot a \cdot 2 \cdot b \cdot 6$	**d.** $0.3 \cdot p \cdot r \cdot p \cdot r \cdot r \cdot 0.2 \cdot r \cdot 10$

12. **a.** Find the value of the expression $10a^4b^2$ when $a = 2$ and $b = 3$.

b. Find the value of the expression $14x^3y^5$ when $x = 2$ and $y = 0$.

13. When you fold a sheet of paper in half, its area is now only 1/2 of the area of the original paper. Let's say you repeat this process, and fold that paper again in half, and again, and again. How many times do you need to fold a sheet of paper in order for the area of the folded piece to be 1/64 of the area of the original?

Puzzle Corner What is the simple value of $\frac{9^6}{9^5}$? There is no need for actual calculations!

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Using Scientific Notation in Calculations, Part 1

Example 1. How many times bigger is one number than another?

You can easily tell that \$60 is three times as much as \$20. But what about \$500 000 and \$150 000 000? Scientific notation makes these types of comparisons very straightforward.

First we write the numbers in scientific notation: \$500 000 = $5 \cdot 10^5$ and \$150 000 000 = $1.5 \cdot 10^8$. Next we divide them, using the quotient rule for exponents: $\frac{1.5 \cdot 10^8}{5 \cdot 10^5} = \frac{1.5}{5} \cdot \frac{10^8}{10^5} = 0.3 \cdot 10^3 = 0.3 \cdot 1000 = 300.$

So, the larger number is 300 times the other. No calculator needed, and in fact, if the exponents had been larger, a regular calculator would not handle the numbers in decimal notation.

Don't confuse the above with simple comparisons where we determine which number is greater, such as $32\,000 < 6 \cdot 10^4$. The above is a *multiplicative* comparison: how many *times* bigger is one number than another?

Do not use a calculator in the problems on this page.

1. The mass of the sun is about $2 \cdot 10^{30}$ kg. The mass of the Earth is about $6 \cdot 10^{24}$ kg. About how many times more massive is the sun than the earth?

2. **a.** How many times bigger is $6 \cdot 10^{-20}$ than $3 \cdot 10^{-30}$?

   b. How many times bigger is $2 \cdot 10^4$ than $8 \cdot 10^{-4}$?

3. The speed of light is approximately $3 \cdot 10^5$ km/s. The distance from earth to sun is approximately 150 million kilometres.

   a. Write the distance in scientific notation.

   b. Now use the two numbers that are in scientific notation, and calculate how long it takes for sunlight to travel from the sun to the earth.

   Give thought to *which* unit of time you will use for the answer; in other words, which unit makes most sense considering the context.

Example 2. When Sheila calculated the value of 70^{23} with a calculator, she got this:

70 ^ 23 =
2.7368747340080916343e+42

The number is given in scientific notation, signifying $2.7368747340080916343 \cdot 10^{42}$. (In reality, the decimal digits would continue for longer; we only see the digits that fit on the calculator screen.)

The letter “e” refers to “exponent”; however this is not referring to the decimal number being raised to the power of 42, but the number 10 being raised to that power.

You may use a calculator for all the rest of the problems in this lesson.

4. A student multiplied two large numbers with a calculator and got this: 1.5E26

   a. What does the answer mean?

   b. What two numbers could she have multiplied?

5. In scientific notation, we use negative exponents for numbers with very small absolute value. Investigate how different calculators show this. *Hint*: divide a small number by a very large number.

6. The speed of light is 299 792 458 m/s. Calculate the distance light travels in a year. This distance is called a *light year*. Give your answer in kilometres, in scientific notation, and with four significant digits.

7. A scientific paper from 2016 estimates that an average 70-kg man has about $3.8 \cdot 10^{13}$ bacteria in his body (most are gut bacteria), and that those bacteria have a mass of about 0.2 kg. What is the average mass of one bacterium in this scenario? (Round your answer considering the significant digits.)

8. A golden eagle can dive at a speed of $2.10 \cdot 10^{7}$ cm per hour. A garden snail is 4600 times slower than the eagle! Find the speed of the garden snail and give it in a reasonable unit, and considering significant digits.

Example 3. The mass of one gold atom is about $3.2696 \cdot 10^{-22}$ grams. How many gold atoms are there in one troy ounce of gold? (1 troy ounce = 31.10348 g)

This is a division problem. We divide 31.10348 grams by the mass of one gold atom: $\dfrac{31.10348 \text{ g}}{3.2696 \cdot 10^{-22} \text{ g}}$.

First off, note that the units "g" cancel out, which is what we would expect, since we expect to get a number without any units (a quantity or "how many").

We will write this quotient in two parts, as $\dfrac{31.10348}{3.2696} \cdot \dfrac{1}{10^{-22}}$, and then work with the two parts separately.

From the calculator, $\dfrac{31.10348}{3.2696} \approx 9.5129$ (five significant digits). The other part, $\dfrac{1}{10^{-22}}$, equals 10^{22}.

The end result is that you need about $9.5129 \cdot 10^{22}$ gold atoms to make one troy ounce of gold.

9. The mass of one gold atom is about $3.2696 \cdot 10^{-22}$ grams.

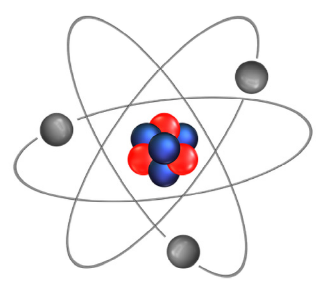
a. What is the approximate mass of a trillion gold atoms?

b. Use the table on the right and give this mass using an appropriate prefix with the unit "gram". For example, the mass of $5 \cdot 10^{-7}$ grams could be given as 0.5 micrograms or as 500 nanograms.

Prefix	Meaning
milli	10^{-3}
micro	10^{-6}
nano	10^{-9}
pico	10^{-12}
femto	10^{-15}
atto	10^{-18}

10. Recall that the nucleus of an atom consists of protons and neutrons, and electrons are very small particles that whiz around the nucleus.

We commonly see images like this, where it looks like the nucleus is maybe about 1/4 of the diameter of the entire atom. But what is the truth of the matter?



Let's look at silicon, for example. The radius of a silicon atom is about 110 picometres. The radius of the *nucleus* of a silicon atom is about 3.6 femtometres.

In the case of silicon, about how many times bigger is the diameter of the entire atom than the diameter of the nucleus?

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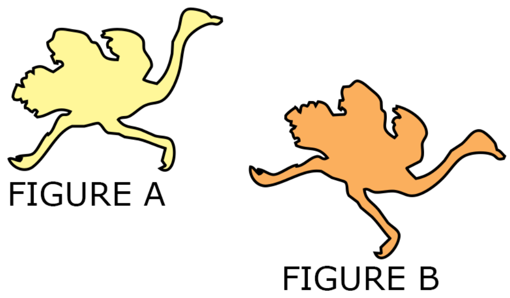
Geometric Transformations and Congruence, Part 1

Two figures are congruent when they are, you might say, identical in the sense that they have the same shape and size (but may be of different colour). We can define congruency as follows:

Two figures are **congruent** if they perfectly match, when one is placed on top of the other.

The figures don't have to be in the same position or orientation. For example, these two figures are congruent — if you rotate and move figure A, you can place it exactly on top of figure B.

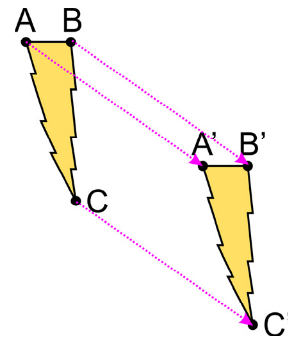


We will now study three geometric **transformations**, or basic ways to move a point, or by extension, a figure, since a figure can be considered to consist of many points.

1. A **translation** of a figure means sliding or moving it a certain distance in a certain direction, without turning or rotating it. The arrows show how three individual points of the figure were moved.

   We say the translation maps point A onto point A' (read "A prime"), point B onto point B', and point C onto point C'.

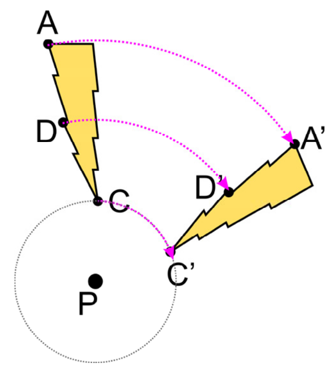
   We also say that point A' is the image of point A under the translation.



2. A **rotation** means turning a figure around a certain point.

   Here, the lightning figure is rotated around point P. Each point of the figure moves in a **circular arc around point P**.

   A rotation is measured in degrees, just like angles are. In this example, the lightning figure was rotated 67 degrees clockwise around point P.

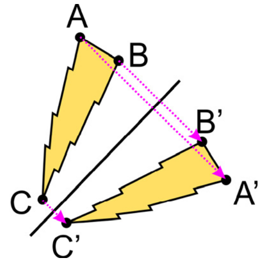


3. A **reflection** across a line means mirroring the figure in that line. You could also say the figure was "flipped".

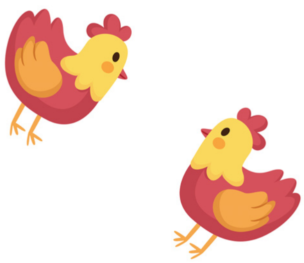
   In a reflection, the distance from each point to the reflection line and the distance of its image to the line are equal (measured along a line segment that is perpendicular to the line).

   For example, the distance from point C to the line equals the distance from point C' to the line.

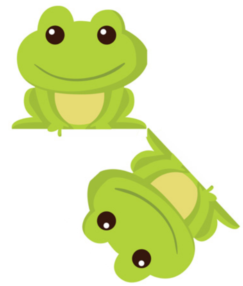
   A reflected figure is congruent to the original.



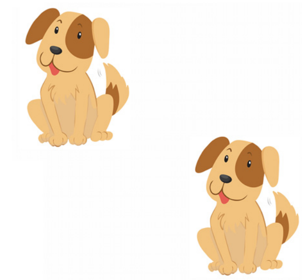
1. Name the transformation that was used to transform the figure on the left to the figure on the right.



a.



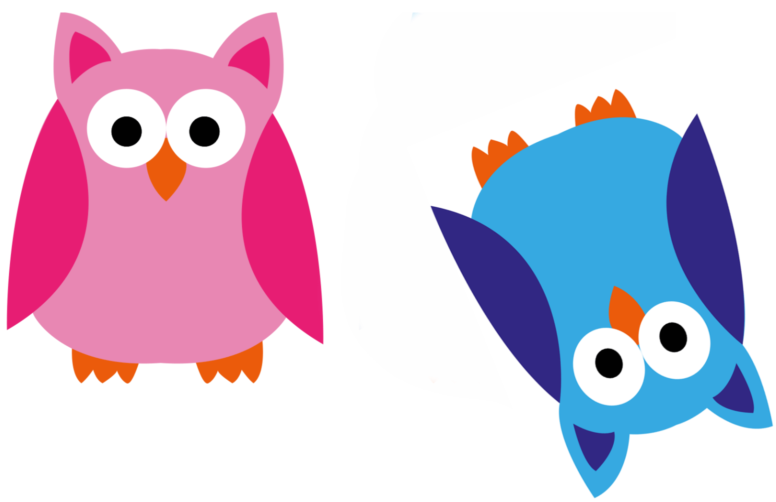
b.



c.

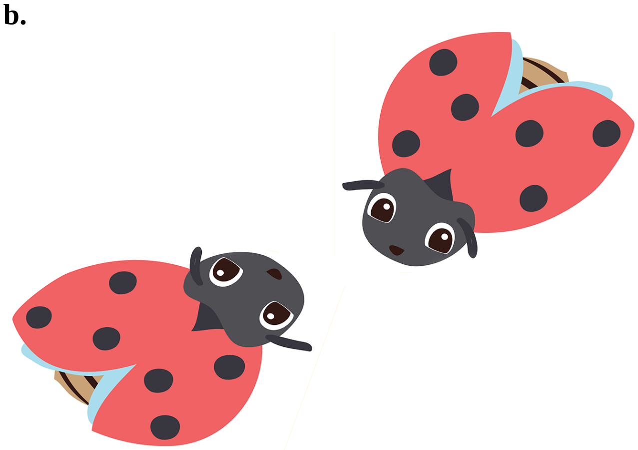
In continuation, we will explore geometric transformations and how they relate to congruence with the help of tracing paper (patty paper) or a transparency.

2. Use tracing paper to determine whether the two figures are congruent. You may move, turn, and/or flip the tracing paper. First, copy the outline of **one** figure to the tracing paper. (Note: when checking for congruency, we ignore the colours.)

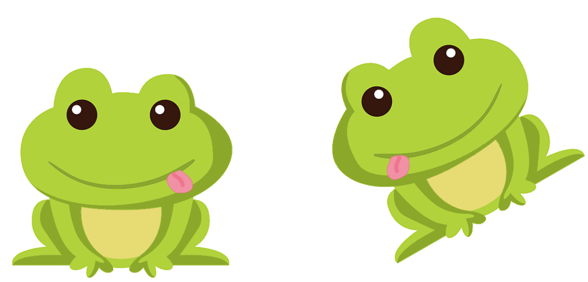


a.

b.

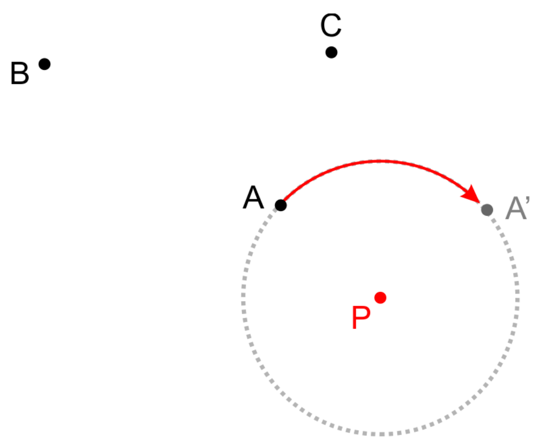


c.



3. The image below shows how point A was mapped to point A' in a rotation. We will now do the same rotation to points B and C using tracing paper. This is how:

i. Put a thumbtack or a pin through the tracing paper at P so that you can turn the paper around P.
ii. Copy points A, B, and C to the paper.
iii. Then rotate the paper around point P so that **point A is mapped to point A'**.
iv. Now, draw the points B' and C'. You can use a pin to mark where these points are (through the tracing paper). Drawing the points with a pencil on the tracing paper may also make a faint mark in the underlying paper. Then remove the tracing paper and draw the points.

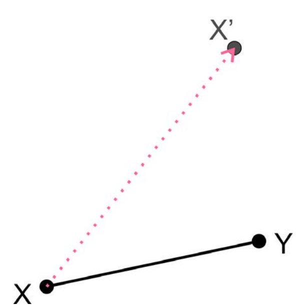


a. Connect A, B, and C with line segments, and also A', B', and C', so that you get two triangles.

b. Measure the side lengths of both triangles. What do you notice?

c. Measure the angles BAC and B'A'C' and also the angles ACB and A'C'B'. What do you notice?

4. Point X' is the image of point X under a translation along the dashed arrow.



a. Sketch the image of point Y in the same translation. Mark it as point Y'.

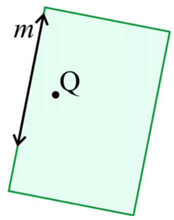
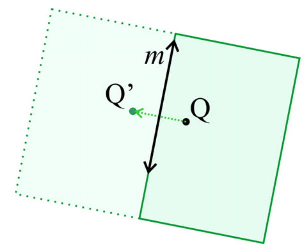
You may optionally do this translation with tracing paper. However, it is difficult to do this accurately.

b. What can we know about the length of the segment $\overline{X'Y'}$? Choose one answer:

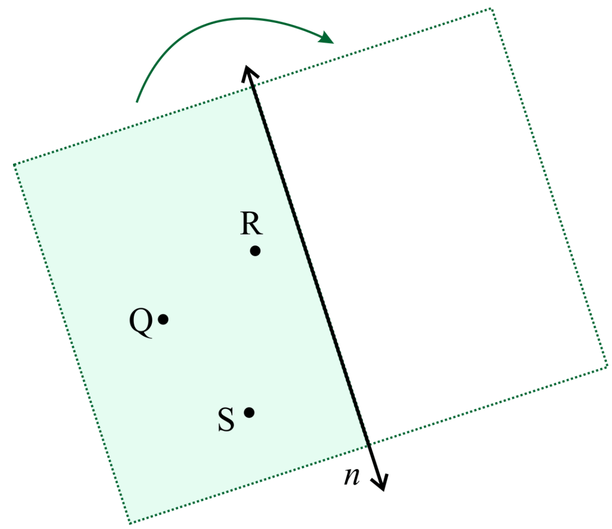
(i) $\overline{XY}$ and $\overline{X'Y'}$ are congruent (have the same length).

(ii) $\overline{XY}$ and $\overline{X'Y}$ are not congruent.

(iii) We cannot know for sure whether $\overline{XY}$ and $\overline{X'Y}$ are congruent or not.

How to reflect a point across a line using tracing paper or a transparency	
Step 1. Align the paper so that one of its edges is along the reflection line *m*. 	**Step 2.** Flip the paper. You can use a pin to mark the image of the point in question. 

5. **a.** Cut out a piece of transparent paper that fits inside the light-coloured rectangle in the image on the right (approximately 3.2 cm by 4.8 cm). Use tracing paper to reflect the points Q, R, and S across line *n*. Label the reflected points as Q', R', and S'.

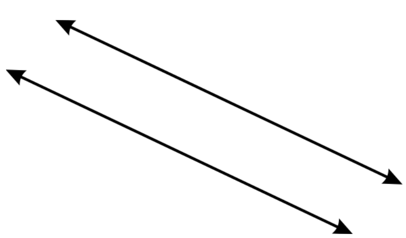


b. Connect the points Q and R, R and S, Q' and R', and R' and S' with line segments.

c. Measure the length of the line segments $\overline{QR}$ and $\overline{Q'R'}$, and also $\overline{RS}$ and $\overline{R'S'}$. What do you notice?

d. Measure also the angles ∠QRS and ∠Q'R'S'. What do you notice?

6. Predict what will happen to parallel lines under translation, rotation, and reflection. You may want to use tracing paper (as needed) to confirm your prediction.



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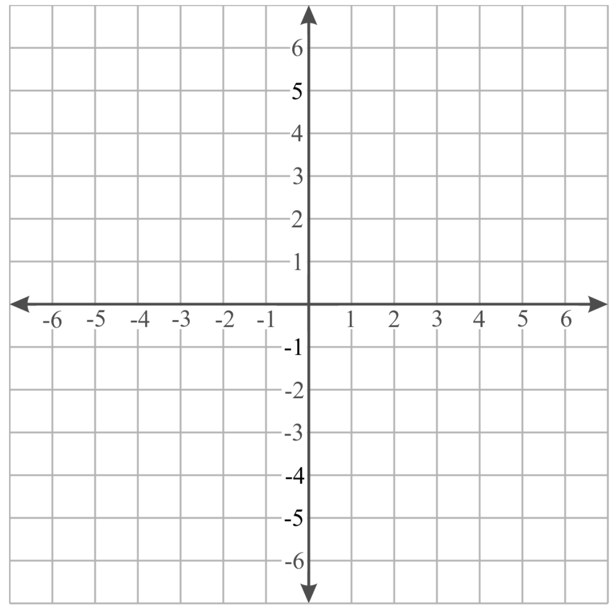
Sequences of Transformations, Part 2

Note: You can use the grid to help you with the following problems, but try to solve them without using it.

1. A triangle with vertices A(1, 2), B(5, 3), and C(4, 1) was first reflected in the *y*-axis and then translated 6 units down and two to the right. What are the coordinates of the vertices of the resulting triangle?

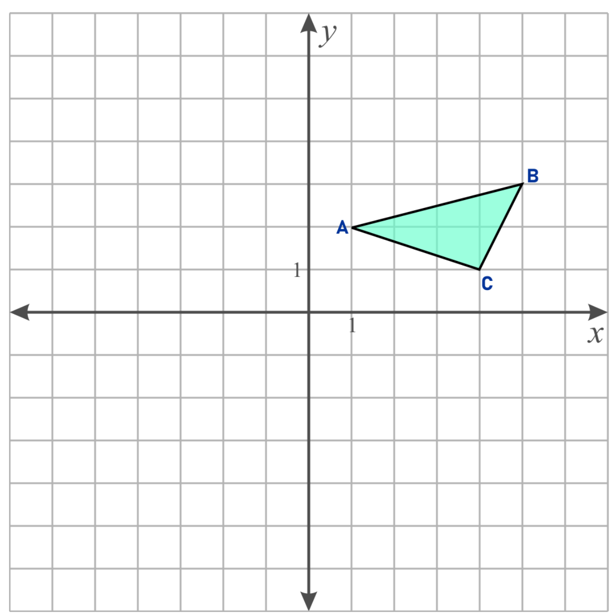


2. Line segment $\overline{AB}$ with A(−2, 4) and B(0, 2) was rotated 180 degrees around the origin and then translated 7 units up and 5 to the left. What are the coordinates of the end points of the line segment after these transformations?

3. A quadrilateral was first rotated around the origin counterclockwise 90 degrees, and then reflected in the *x*-axis. Its vertices are now at points (3, 5), (5, 2), (3, 1), and (2, 2). What were the coordinates of its vertices before these transformations?

4. Triangle ABC is as shown on the right. It will be rotated around the origin counterclockwise 90 degrees, then translated 5 units down, and lastly, rotated once again around the origin counterclockwise 90 degrees.

   Ashley claims that the transformed triangle's vertices are at (5, −2), (1, −3), and (2, −1). Is she correct? Explain.

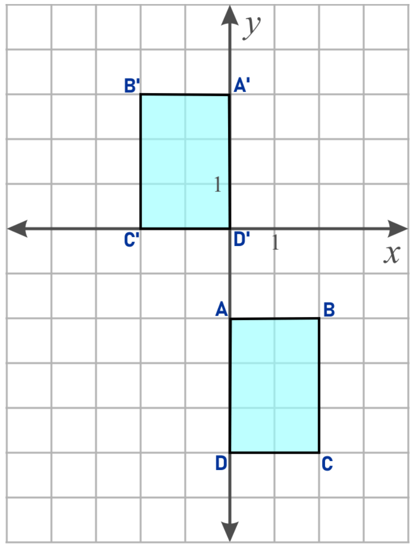


5. Greg says that the two rectangles are congruent because you can reflect rectangle ABCD in the *y*-axis and then move it five units up to map it onto rectangle A'B'C'D'.

   Jenny says that's too complicated; you can simply translate rectangle ABCD five units up and two units to the left, and that does the job.

   Who is correct, or are both correct? Why?

   (Hint: Note the vertices carefully.)



6. A quadrilateral with vertices H(−5, 2), I(−4, 4), J(−2, 4), and K(−4, 1) is reflected in the horizontal line $y = 1$, and then rotated around the origin 180 degrees. Find the coordinates of the transformed figure.

7. Triangle PQR underwent a translation, then a reflection. Study the coordinates to find out the details about each transformation, then fill in the missing coordinates.

Original figure	Translation	Reflection
P(−5, −2)	P'(−6, 3)	P"(4, 3)
Q(−3, −2)	Q'(−4, 3)	Q"(2, 3)
R(−4, 1)	R'(_____ , _____)	R"(_____ , _____)

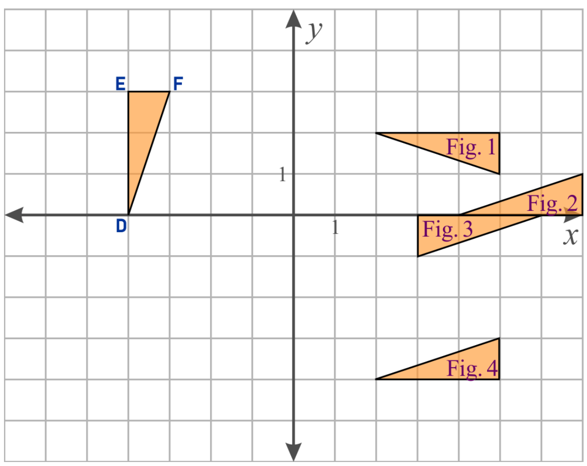
8. Which of the figures 1, 2, 3, or 4 is the image of triangle DEF when it undergoes the following sequence of transformations?

   1. Rotation 90° clockwise around D;

   2. Rotation 180° around the origin;

   3. Reflection in the *x*-axis;

   4. Translation two units to the right.



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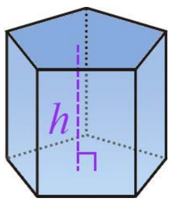
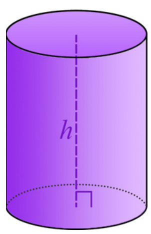
Volume of Prisms and Cylinders



A pentagonal prism. The dashed line marks the height (*h*).



A circular cylinder

In geometry, a **prism** is a 3-dimensional solid with two identical parallel polygon faces, known as its **bases**, and the rest of the faces are rectangles (or parallelograms, if the prism is oblique, or "slanted"). In this book we will only deal with right prisms (ones that are not slanted).

Note that a box with a rectangular base is a prism; in geometry we call it a *rectangular* prism.

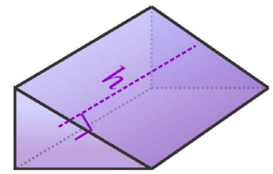
A **cylinder** is similar to a prism, but its bases are circles or ellipses, and it has just one other face that is wrapped around the bases.

Both prisms and cylinders are named after the polygon or shape at their base.

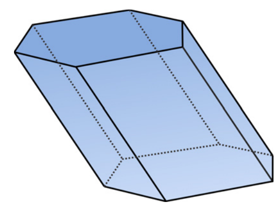
The **height** of both prisms and cylinders is the length of the line segment drawn from the top face to the bottom face that is perpendicular to both faces.

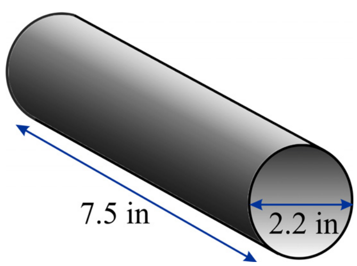
The **volume** of prisms and cylinders is calculated in the same way: we simply **multiply the area of the base** (A_b) **by the height** (*h*).

The formula is: $\mathbf{V = A_b}\boldsymbol{h}$.



A triangular prism. Note that its bottom face is facing the viewer. The dashed line marks the height (*h*).


An oblique hexagonal prism



Example 1. Calculate the volume of this cylinder.

This cylinder is "lying down." Imagine standing it up to see what the top and bottom faces are. Its top and bottom faces are circles.

Notice that we are given the *diameter* of the circle, but to calculate the area of a circle, we need to use the *radius*, which is 1.1 inches.

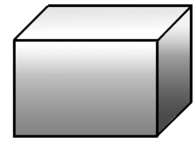
Using the formula for the area of a circle, $A = \pi r^2$, we get that the area of the bottom face is $\pi \cdot (1.1 \text{ in})^2$. The height is 7.5 in.
The volume is the product of the two: $V = \pi \cdot (1.1 \text{ in})^2 \cdot 7.5 \text{ in} \approx \underline{28.5 \text{ in}^3}$.

You can use a calculator in all the problems in this lesson.

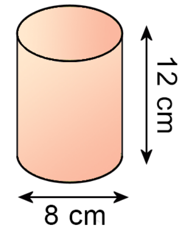
1. You have learned to calculate the volume of a box (a rectangular prism) by multiplying its width, depth, and height (its three dimensions).

   Does the formula $\mathbf{V = A_b}\boldsymbol{h}$ also apply to boxes?

   Why or why not?

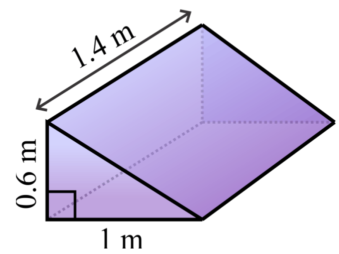


2. The bottom face of this cylinder is a circle with a diameter of 8 cm. Its height is 12 cm. Find its volume to the nearest ten cubic centimetres.



3. **a.** What is this shape called? If you are unsure, ask yourself: what are the two identical parallel faces?

   b. Calculate its volume.



4. The Fernandez family has three cylindrical water tanks, of different sizes. The first one has a diameter of 1.52 m and a height of 1.8 m. The second and third have a diameter of 2.4 m and a height of 3.0 m.

   a. Calculate their total volume in cubic metres.

   b. The family uses 450 litres of water per day, on average. If the water tanks are full, how many days of water supply do they provide for the family? Note: One cubic metre = 1000 litres.

5. Find a drinking cup or a mug with a cylindrical shape. Most drinking glasses taper down towards the bottom so they don't work. Look for one whose bottom and top faces are congruent circles.

   a. Measure the mug, and calculate its volume in cubic centimetres.

   b. Measure its volume now in millilitres, using a measuring cup, and compare to what you got above. Remember that 1 ml = 1 cm^3.

If the results are far apart, check your measurements. Check also whether your measuring cup is accurate (often they are not).

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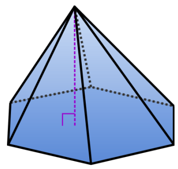
Volume of Pyramids and Cones

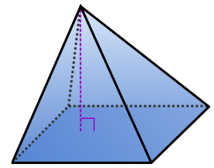
A pyramid is a solid that has some polygon as a base. Its other faces are triangles that meet at the top vertex of the pyramid.



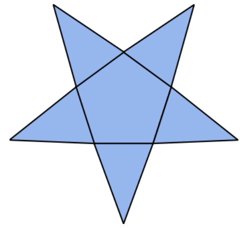
A hexagonal pyramid

Just like prisms, pyramids also are named after the polygon at their base. A rectangular pyramid has a rectangle as its base, a triangular pyramid has a triangle as its base, a pentagonal pyramid has a pentagon as its base, and so on.

The *height* or *altitude* of a pyramid is the length of the line segment drawn from the top vertex to the base so that it is perpendicular to the base. We need the height in calculating the volume of pyramids.



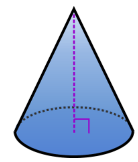
A square pyramid

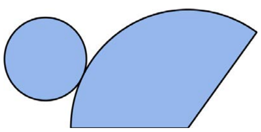


Can you tell what kind of pyramid the net on the right belongs to?

You can find the answer below this blue box, but think first!

A cone is similar to a pyramid, but it has a rounded shape as its base. The cone on the right is a circular cone. And similarly with pyramids, a cone has a *height*: a line drawn from the vertex that is perpendicular to the base.



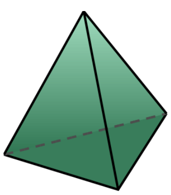


The net of a cone has two parts: a circle (the base), and a sector (a part) of a circle, which is the other face of the cone — the one you wrap around the base.

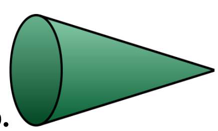
Note: The net above is for a pentagonal pyramid: it has a pentagon as a base, and triangles as the other faces.

You may use a calculator in all problems in this lesson.

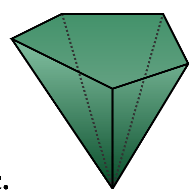
1. Name the solids.



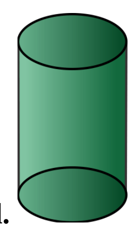
a.



b.

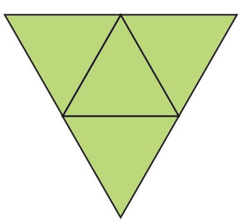


c.

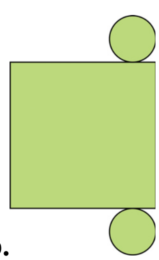


d.

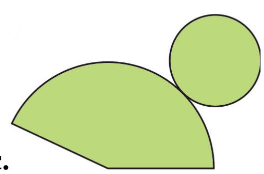
2. Name the solids that can be constructed from these nets.



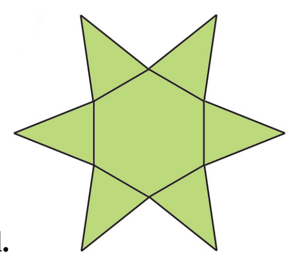
a.



b.



c.

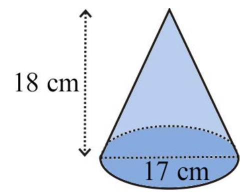


d.

The **volume** of all pyramids and cones is calculated in the same way: It is one-third of the area of the base (A_b) multiplied by the height (h). It doesn't matter whether the cone or the pyramid is slanted or upright; the formula works in either case.

As a formula, we write $V = \frac{1}{3} A_b h$.

Example 1. Calculate the volume of the cone to the nearest ten cubic centimetres.



First, let's find out the area of the base. It is a circle with a radius of 8.5 cm, so its area is $A_b = \pi \cdot (8.5\text{ cm})^2 \approx 226.865\text{ cm}^2$.

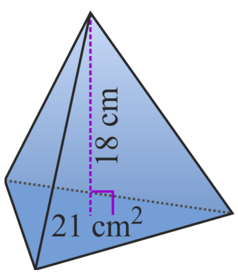
Note: don't round your intermediate answers a lot. Keep a few extra digits just to be safe. Rounding to the nearest ten should only happen in the final step.

Now, the volume. Using the formula, we get $V = \frac{1}{3} A_b h = \frac{1}{3} \cdot 226.865\text{ cm}^2 \cdot 18\text{ cm}$

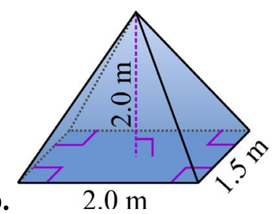
$= 1361.19\text{ cm}^3$ or approximately <u>1360 cm</u>^{3}.

3. Calculate the volumes of these solids. Note: the cones are circular (have a circle as their base).

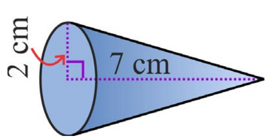
a.



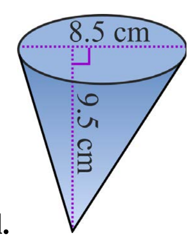
b.



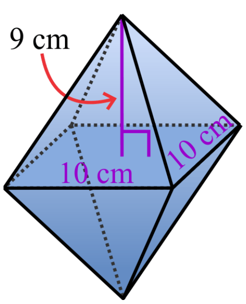
c.



d.

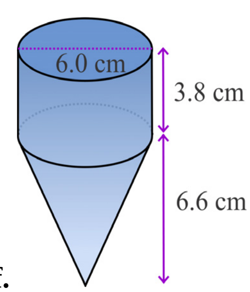


e.



an octahedron:
two square pyramids

f.



a circular cone and
a circular cylinder

Notice something similar about the two formulas for volume that we have studied:

Volume of a prism or cylinder: $\mathbf{V = A_b}\boldsymbol{h}$.

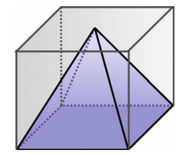
Volume of a pyramid or cone: $\mathbf{V = (1/3)A_b}\boldsymbol{h}$.

This means that if we take a pyramid and a prism with the same base and same height, the volume of the pyramid is exactly one-third of the volume of the prism. The same is true of a cone and a cylinder with the same base and same height.

This relationship might remind you of something similar concerning areas: the area of a triangle is always 1/2 of the area of a parallelogram with the same base and height!

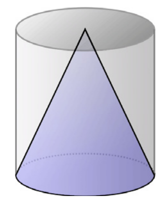
A pyramid inside a box:
(The pyramid and the prism share the same base and the same height.)

The volume of the pyramid is 1/3 of the volume of the box.

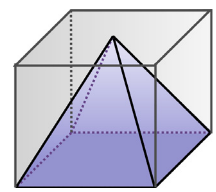


A cone inside a cylinder:
(The cone and the cylinder share the same base and the same height.)

The volume of the cone is 1/3 of the volume of the cylinder.



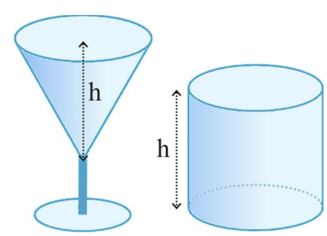
4. A cube with a volume of 27 000 cm^3 has a square pyramid inside it so that the base of the pyramid is the same as the base of the cube, and its vertex touches the top of the cube.



   a. What is the volume of the pyramid?

   b. How long is the side of the cube?
   Hint: Guess and check.

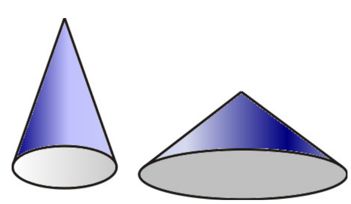
5. A conical and a cylindrical drinking glass have the same height. The top of the conical glass and the top of the cylinder are congruent circles.



   a. What percent of the volume of the cylindrical glass is the volume of the conical glass?

   b. Assume the conical glass is full of water, and the cylindrical glass is 2/3 full. Now what percent is the volume of the water in the conical glass as compared to the volume of the water in the cylindrical glass?

6. The taller "party-hat" has a bottom diameter of 10 cm and a height of 25 cm. The diameter of the second hat is twice the diameter of the first, and its height is half the height of the first hat. Would that mean that their volumes are equal?



   Find out by calculating their volumes.

   What is the simple relationship between their volumes?

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Algebra Terms
For Reference

Expressions in mathematics consist of:

- numbers;
- mathematical operations (+ , − , · , ÷ , exponents);
- and letter variables, such as *x*, *y*, *a*, T, and so on.

Note: Expressions do *not* have an equals sign!

Examples of expressions:

$\frac{3}{5}x^2 - 3x + 5$ $\quad$ 5 $\quad$ $\left(\frac{3x}{y^2}\right)^2$

$T - 29$ $\quad$ $2^x - 5^y$

An **equation** has two expressions separated by an equals sign:

(expression 1) = **(expression 2)**

Examples of equations:

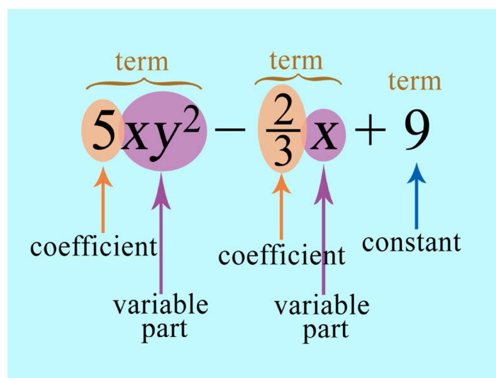
$0 = 0$ $\quad$ $2(z - 9) = -z^2$

$9 = -8$ (a false equation) $\quad$ $\frac{x + 3}{2} = -1.5$

A **term** is an expression that consists of numbers and/or variables that are *multiplied*. For example, $7x$ is a term and so is $0.6mn^2$.

A single number or a single variable is also a term. If the term is a single number, such as 4.5 or ¾, we call it a **constant**.

In the expression on the right, we have three terms: $5xy^2$, $\frac{2}{3}x$, and 9, that are separated by subtraction and addition.



If a term is not a single number, then it has a **variable part** and a **coefficient**.

- The coefficient is the single number by which the variable or variables are multiplied.
- The variable part consists of the variables and their exponents.

For example, in $4.3ab$, 4.3 is the coefficient, and *ab* is the variable part.

<u>Note</u>: a term that consists of variables only still has a coefficient: it is one. For example, the coefficient of the term x^3 is one, because you can write x^3 as $1 \cdot x^3$.

Example. Is $s - 5$ a term? No, it is not since it contains subtraction. Instead, $s - 5$ is an expression consisting of two terms, *s* and 5, separated by subtraction.

1. Write the expression based on the clues.

- It has four terms.
- The constant term is the square of the third smallest prime.
- The variable parts of the variable terms are *ab*, a^2, and *a*, respectively.
- The coefficients of the variable terms are the three consecutive integers with a sum of 21.
- The first two terms are separated by subtraction, the rest by addition.

Review: Integer Addition and Subtraction

Integers consist of the counting numbers (1, 2, 3, 4, ...), zero, and the negative counterparts of the counting numbers (−1, −2, −3, −4, ...). So, the set of integers is {..., −4, −3, −2, −1, 0, 1, 2, 3, 4, ...}.

An **absolute value** of an integer is its distance from zero, and is marked with two vertical lines. For example, $|2| = 2$ and $|-18| = 18$.

We obtain the **opposite** or **negation** of an integer by changing its sign from positive to negative, or vice versa. For example, the opposite or negation of 17 is −17. The opposite of −4 is 4.
We can use the negative sign "−" to signify this: −(−5) means the opposite of −5, which is 5.

To **add several negative integers**, simply add their absolute values and write the answer as negative.

Example 1. To find the sum −8 + (−3) + (−7) + (−11), add 8 + 3 + 7 + 11 = 29.
The value of the original sum is −29.

To **add a negative and a positive integer**, find the difference in their absolute values. The integer with the bigger absolute value determines the sign of the final answer.

Example 2. In the sum −9 + 11, the absolute values of the two integers are 9 and 11. Their difference is 11 − 9 = 2. This means the answer is either 2 or −2. To determine which, check the sign of the integer with the larger absolute value. In our case it is 11 (which is positive), so the answer is 2 (and not −2).

Example 3. In the sum 7 + (−12), the absolute values of the two integers are 7 and 12. Their difference is 12 − 7 = 5. This means the answer is either 5 or −5. To determine which, check the sign of the integer with the larger absolute value. Here it is −12 which is negative, so the answer is −5 (and not 5).

So, this is the mechanical rule, but you don't have to use it if you have learned other methods, such as visualizing a number line.

To **add several integers** where some are negative, some positive, first calculate the partial sums of all the negative integers and of all the positive ones. Lastly add those sums.

Example 4. −8 + 12 + (−9) + (−1) + 5 + (−6) = ?

Positives: 12 + 5 = 17

Negatives: −8 + (−9) + (−1) + (−6) = −24

Total: 17 + (−24) = **−7**

1. Add.

a. (−4) + 8 =	**b.** 15 + (−25) =	**c.** −12 + 6 =	**d.** −11 + (−32) =

e. −12 + (−2) + (−5) =	**f.** 6 + (−1) + (−5) + 2 =	**g.** −7 + 10 + (−6) + 1 =

h. −11 + (−2) + 7 + (−5) + 4 + (−3) =	**i.** −6 + (−5) + 8 + (−12) + 24 + 1 =

To **subtract two integers**, you can often think with the help of the number line model. For example, you can visualize 2 − 6 as starting at 2, and moving 6 steps to the left on the number line.

Mathematicians actually define the subtraction of two numbers, $a - b$, as the sum of a and the opposite of b.

In symbols: $a - b = a + (-b)$

In other words, to subtract an integer, change the subtraction to an addition of the opposite number. From this definition it also follows that $a - (-b)$ simplifies to $a + b$.

(Why? In $a - (-b)$ we subtract $-b$, and the opposite of $-b$ is b. Instead of subtracting $(-b)$, you add *its opposite*, or b.)

Example 5.	5 − 7 ↓ 5 + (−7) = −2	−6 − 8 ↓ −6 + (−8) = −14	2 − (−4) ↓ 2 + 4 = 6	−3 − (−9) ↓ −3 + 9 = 6

2. Write each subtraction as an addition, and solve.

a. 8 − (−6) ↓ 8 + 6 = ______	**b.** −9 − (−14) ↓ ____ + ____ = ______	**c.** −21 − 8 ↓ ____ + ____ = ______	**d.** 3 − 15 ↓ ____ + ____ = ______

3. Subtract.

a. 2 − 9 =	**b.** −2 − 9 =	**c.** −2 − (−9) =	**d.** 2 − (−9) =
e. −7 − 4 =	**f.** −7 − (−4) =	**g.** 7 − (−4) =	**h.** 4 − 7 =

4. Can any addition be changed to a subtraction? See if you can find matching subtractions for these additions.

a. 7 + (−10) ____ − ____ = _____	**b.** −2 + (−1) ____ − ____ = ______	**c.** 14 + 3 ____ − ____ = ______	**d.** −8 + 5 ____ − ____ = ______

So, any subtraction can be written as an addition. The converse is also true: any addition can be written as a subtraction. For example, the sum 5 + 4 can be written as 5 − (−4), and the sum 2 + (−13) can be written as the subtraction 2 − 13.

In symbols, $c + d = c - (-d)$. Instead of adding d, you subtract the opposite (or negation) of d.

However, since this usually does not simplify the calculation, it does not get used often.

5. Solve, working in order from left to right.

a. −2 − 5 + 6 =	**b.** 7 + (−12) − 5 − 8 =	**c.** −1 + 9 − 14 + 7 =
d. −8 − 12 − 5 + 9 =	**e.** 2 − (−12) − 10 − (−3) =	**f.** −21 − 13 + 8 − (−5) =

Adding negative fractions

Example 6. Add $\frac{2}{5} + \left(-\frac{3}{7}\right)$. We could use the regular process for integer addition: figure out which fraction has a larger absolute value, then subtract the smaller absolute value from the larger one, and so on. But the easier way is this: Simply add the fractions normally and treat the negative fraction $-\frac{3}{7}$ as $\frac{-3}{7}$.

You will end up with an *integer addition* in the numerator. See the full solution on the right.

$$\frac{2}{5} + \left(-\frac{3}{7}\right)$$

$$\frac{2}{5} + \frac{-3}{7}$$

$$\frac{14}{35} + \frac{-15}{35}$$

$$\frac{14 + (-15)}{35} = \frac{-1}{35} = -\frac{1}{35}$$

6. Add and subtract.

a. $\frac{2}{7} + \left(-\frac{3}{4}\right)$	**b.** $\frac{1}{8} + \left(-\frac{1}{2}\right) + \left(-\frac{3}{4}\right)$
c. $-\frac{5}{6} + \frac{2}{9}$	**d.** $-\frac{2}{3} + \frac{2}{9} - \frac{1}{6}$
e. $-\frac{7}{8} + \left(-\frac{1}{10}\right)$	**f.** $\frac{1}{6} - \left(-\frac{7}{8}\right)$

Equations Review, Part 1

An **equation** consists of two expressions, separated by an equals sign:

expression 1 = **expression 2**

For example, $40 = w + 32$ is an equation, and so is $2 = 5$, the latter being a *false* equation.

A **solution** or a **root** to an equation is a value of the unknown that makes the equation *true*; in other words, makes the two expressions on both sides to have the same value.

Example 1. Is 20 a root to the equation $11 = \frac{1}{2}x + 3$?

To check that, we substitute 20 in place of *x* and check whether the two sides of the equation have the same value:

$$11 \stackrel{?}{=} \frac{1}{2}(20) + 3$$

$$11 \neq 10 + 3$$

No, 20 does not fulfil this equation, so it is not a root.

1. **a.** Is 2 a root to the equation $\frac{3x^2 - 7}{5} = x$? Explain.

   b. Is −90 a root to the equation $\frac{2}{3}y + 11 = -49$? Explain.

2. Without solving the equation, check whether $x = -3$ is a solution to the equation $x + 4x + 6x - 8 = -5(x + 8)$. Before you start, think: would you be allowed to simplify the left side of the equation?

3. Write three different equations with the solution of $x = -5$.

4. If $2w + 6 = 50$ and $3w - 15 = 51$, then does $2w + 6$ equal $3w - 15$? Justify your reasoning.

To solve an equation, we perform the same mathematical operation (add, subtract, multiply, divide) to *both* sides of the equation. Notice that in that process, the two sides of the equation remain equal, even though the expressions themselves, on both sides, change!

Note: We can **mark the operation to be done to both sides** either below each line of the solution or in the right margin, after a vertical line. I prefer marking it in the right margin, because that is how I was taught in school in Finland, but you can go with whatever you or your teacher prefers.

Example 2.

$$7w + 8 = 43$$
$$\underline{\quad -8 \quad -8}$$
$$7w = 35$$
$$\frac{7w}{7} = \frac{35}{7}$$
$$w = 5$$

Check: $7(5) + 8 \overset{?}{=} 43$

$43 = 43$ ✓

Example 3.

$$-14 = \frac{1}{5}a$$
$$5(-14) = 5 \cdot \frac{1}{5}a$$
$$-70 = a$$
$$a = -70$$

Check: $-14 \overset{?}{=} \frac{1}{5}(-70)$

$-14 = -14$ ✓

Example 4.

$$-18 - x = 24 + 8$$
$$-18 - x = 32 \qquad | + 18$$
$$-x = 50 \qquad | \div (-1)$$
$$x = -50$$

Check: $-18 - (-50) \overset{?}{=} 24 + 8$

$-18 + 50 \overset{?}{=} 32$

$32 = 32$ ✓

5. See Derek's solution on the right.

a. Check whether $x = 14$ is truly a root.

b. If not, correct the error in his solution.

$$10 - x = 24$$
$$\underline{\quad -10 \quad -10}$$
$$x = 14$$

6. Solve the equations. Check your solutions.

a. $-p = 12 - 34$	**b.** $78 - x = -8$	**c.** $-2 - 7 = -3z$
d. $\frac{y}{-4} = -22$	**e.** $10x = -40 - 5$	**f.** $2.1 - x = -6.7$

Remember, our goal is to **isolate the unknown** (have it alone on one side).

This example shows a typical two-step equation. On the side of the unknown (left), there is both a multiplication by 8 and an addition of 7. We need to undo both of those operations, in two steps.

Example 5.

$$8x + 7 = -5 \quad | -7$$
$$8x = -12 \quad | \div 8$$
$$x = -12/8$$
$$x = -1\tfrac{1}{2}$$

Check:

$$8 \cdot (-1\tfrac{1}{2}) + 7 \stackrel{?}{=} -5$$
$$-12 + 7 \stackrel{?}{=} -5$$
$$-5 = -5 \checkmark$$

7. Solve the equations. Check your solutions.

a. $5x + 2 = 67$	**b.** $-3y + 2 = 71$	**c.** $25 - 3w = 17$
d. $-34 = 2x - 11$	**e.** $-98 = -8z - 2$	**f.** $-8 - 4z = 10$

8. The solution on the right shows a common student error. We can verify the root is *not* ½ by substituting it to the original equation:

$$14 - 80(\tfrac{1}{2}) \stackrel{?}{=} 54$$

$$14 - 40 \stackrel{?}{=} 54$$

$$-26 \neq 54$$

What is the error? Correct it.

$$14 - 80x = 54$$
$$-14 \quad -14$$
$$80x = 40$$
$$\frac{80x}{80} = \frac{40}{80}$$
$$x = \tfrac{1}{2}$$

9. Use these equations for more practice, as necessary.

a. $-5 + 15 = -6w$	**b.** $6 = \dfrac{d}{-1.1}$	**c.** $\dfrac{a}{5} = -1.2 + (-3.1)$
d. $56 - 5x = 28$	**e.** $-35 = -4q + 2$	**f.** $-150 + 30w = 60$
g. $13.5 - 2y = 7$	**h.** $7.8 - 16.2 = \dfrac{x}{7}$	**i.** $-55 = -6w - 13$

The Distributive Property

The **distributive property** states that we can distribute multiplication over addition: $a(b + c) = ab + ac$.

It also applies with subtraction: $a(b - c) = ab - ac$.

Example 1. Each term needs to be multiplied by the factor in front of the parentheses.
In other words, below, you need to "take the 5 through" onto *everything* inside the parentheses:

$5(x + y + 6) = 5x + 5y + 5 \cdot 6$, which simplifies to $5x + 5y + 30$

Example 2. Be careful with negative numbers and with subtraction. Compare the two examples on the right.

In both, we take −2 through the parentheses. Notice how the "+" sign in $x + 1$ and the "−" sign in $x - 1$ is preserved.

(For clarity, I've written the equivalent expressions one under the other, instead of as one long horizontal line.)

$-2(x + 1)$
$-2 \cdot x + (-2) \cdot 1$
$-2x + (-2)$
$-2x - 2$

$-2(x - 1)$
$-2 \cdot x - (-2) \cdot 1$
$-2x - (-2)$
$-2x + 2$

1. The two multiplications on the right show a common student error. What is that error?

$4(a + b + 3) \rightarrow 4a + 4b + 3$ and $7(2x - 9) \rightarrow 14x - 9$

2. Multiply using the distributive property. Write your answer below the original. Compare the problems. Be careful with negative numbers, and be on the lookout for a shortcut.

a. $-2(x + 9)$	**b.** $-2(x - 9)$	**c.** $-3(5x + 8)$	**d.** $-3(5x - 8)$	**e.** $-5(2x + 7)$	**f.** $-5(2x - 7)$

Example 3. Study these examples of a quicker way of removing the parentheses.

If you're unsure of this, you can use the longer route as shown in example 2.

$-5(2x - 6)$
$-5(2x) - 5(-6)$
$-10x + 30$

$-7(-3x + 6y - 4)$
$-7(-3x) - 7(6y) - 7(-4)$
$21x - 42y + 28$

3. Multiply using the distributive property. Compare the problems.

a. $-2(x + y - 9)$	**b.** $-2(x - y + 9)$	**c.** $-3(9 + 2y)$	**d.** $-3(9 - 2y)$

Example 4. To simplify the expression $-(x + 8)$, think of it as $-1(x + 8)$ (*Why?). We can now use the distributive property →

In a nutshell, $-(x + 8) = -x - 8$. It is as if we take the minus sign through the parentheses, and distribute it to each individual term. This *changes the sign* of each individual term inside the parentheses.

In yet other words, $-(x + 8)$ is the opposite of the expression $x + 8$, and it is obtained by changing each individual term to its opposite.

$-1(x + 8)$

$-1(x) + (-1)(8)$

$-x - 8$

4. Find the opposites of the expressions.

a. $-(y + 3)$	**b.** $-(y - 3)$	**c.** $-(6 - a + 2b)$	**d.** $-(-6 + a - 2b)$

5. Multiply using the distributive property.

a. $-0.9(a - 2b + 7)$	**b.** $-(-x + 4 - 5y)$	**c.** $-(-7w + 0.5)$	**d.** $20(-0.4 + 0.5x)$
e. $\frac{1}{4}(g - 2h + 4)$	**f.** $32(v + \frac{1}{4}w - \frac{1}{8})$	**g.** $-\frac{2}{5}(10p - 20q)$	**h.** $-100(-0.01 + 0.1z)$

Example 5. The expression $(8 + 2x)(7)$ means the same as $(8 + 2x) \cdot 7$, and it is equivalent to $7(8 + 2x)$.

Why? Because multiplication is commutative (can be done in any order). Here, $8 + 2x$ and 7 are the two factors being multiplied, and it doesn't matter in which order we multiply them.

6. Simplify.

a. $(8 + 2x) \cdot 7$	**b.** $(2y - 8 - 3x)(5)$	**c.** $(0.6x - 1.4)(-10)$
d. $(60s - 12 + 39t) \cdot \frac{1}{3}$	**e.** $(12w - 4) \cdot 6 + 15w$	**f.** $-\frac{3}{5}(25x - 45y) + 6x$

*Negative one times a number is equal to the opposite of the number: $-1 \cdot a = -a$. Considering $(x - 6)$ as a *single* number, we get $-1(x - 6) = -(x - 6)$.

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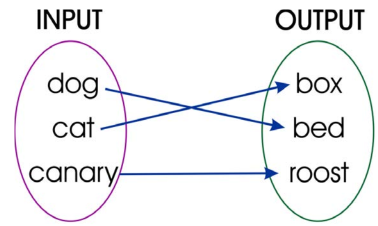
Functions

A **function** is a rule or a relationship between two sets that assigns **exactly one output for each input.** We also use the word **mapping** for a function.

Example 1. The illustration below shows a simple function that maps each animal to its favourite sleeping place.

Each animal has a sleeping place, and only one, so this is a function.

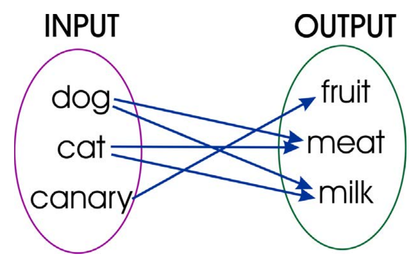


Example 2. The table lists the name of seven children, and the month when each child has their birthday. Notice that several of them have their birthday in December. Is this a function?

Input	Allie	Julie	Danny	Juan	Pete	Bob	Samantha
Output	September	December	December	June	August	December	February

Yes. The definition only requires that there has to be exactly one output for each input; **the outputs don't have to be unique.**

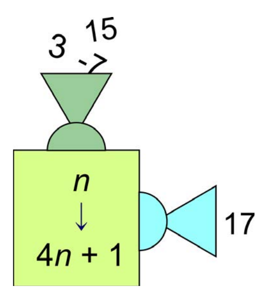
1. The relationship shown on the right is *not* a function. Why?



2. A function machine "ingests" a number (the input) and "spits out" another (the output) based on some rule. This function machine turns any number n into $4n + 1$.

   a. Number −7 is just going in. What will be the output?

   b. Number 17 just came out. What was the input?



3. Potatoes cost $3 per kilogram. Fill in the tables #1 and #2.

   Does each table represent a function? Explain.

#1

(Input) **Weight**	(Output) **Cost**
1 kg	$3
2 kg	
3 kg	
5 kg	
12 kg	

#2

(Input) **Cost**	(Output) **Weight**
$12	
$30	
$48	
$72	
$90	

4. The table lists seven children, and each child's favourite colour.

Input	Allie	Julie	Danny	Juan	Pete	Bob	Samantha
Output	pink and blue	blue	grey	yellow	blue and red	?	purple

Is this a function? If not, change it in some manner(s) so it *is* a function.

5. T is a function that maps the name of a month to the number of days in it.

a. Create a depiction of T using a diagram like in example 1.

b. If you reverse the inputs and outputs, is the resulting relationship a function? Explain.

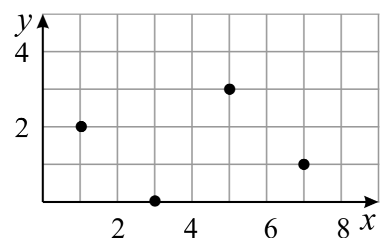
If the inputs and outputs are numbers, we can plot **a graph of the function** in the coordinate grid. Each input-output pair is viewed as an ordered pair (a single point).

We also use the terms "independent variable" for the input, and "dependent variable" for the output.

Example 3. Let F be the function (1, 2), (3, 0), (5, 3), (7, 1).

Note: A function *can* be given as a list of ordered pairs.

The image on the right is the plot of F; yet the plot is *not* F. The function F is the specific list of inputs and outputs, or the relationship itself.



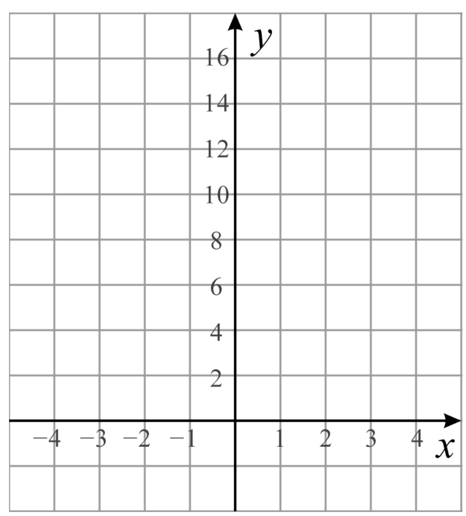
6. Let G be the function that maps each integer from −4 to 4 to its square minus one.

a. Fill in the table, listing the ordered pairs of G.

Input (x)	−4	−3	−2						
Output (y)	15								

b. Make a plot of G.

c. If you reversed the inputs and the outputs, would the relationship still be a function? Explain.



Example 4. Mary bicycled from her home to a friend's house. The table shows the distance (d) Mary had covered at specific amounts of time (t).

Input (t)	5 min	10 min	12 min	13 min	15 min	18 min	20 min	22 min
Output (d)	0.8 km	1.5 km	1.9 km	1.9 km	1.9 km	2.4 km	2.7 km	3 km

We say that **distance is a function of time.** The output variable, or the dependent variable, is always said to be a function of the input (or independent) variable. This means that for each moment of time (input) there is a specific distance she has travelled (output).

Is it true in reverse? Is *time* a function of *distance*?

This means we consider distance as the input, and time as the output. If yes, then for each distance (input), there is exactly one time (output). Is that so in this case?

7. Is Age a function of Name? Explain.

Is Name a function of Age? Explain.

Name	Age
FenFen	14
Larry	15
Pierre	13
Sam	12
Amy	14

Age	Name
14	FenFen
15	Larry
13	Pierre
12	Sam
14	Amy

8. Choose the relationships that are functions.

(1)

Rainfall (mm)	2	0	0	5	0	13	0
Day of month	6	7	8	9	10	11	12

(2) Let S be a rule that takes any number x as input, and gives $4x + 1$ as output.

(3) Input is a zip code, output is a person that lives there.

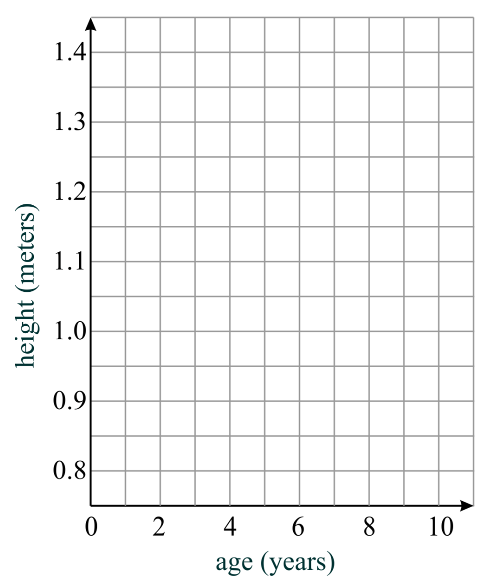
(4) Input is a person's first name, output is their bank account number.

9. Plot the following points that give the age (in years) and the height (in metres) of various children.

(2, 0.8) (5, 1.05) (10, 1.40) (9, 1.31) (6, 1.17) (5, 1.09)

a. Is this a function? Explain.

b. What is the independent variable? The dependent variable?



(Optional content; beyond the CSS)

The **domain** of a function is the set of inputs. The **range** of a function is the set of outputs.

Let's go back to example 3, where we had kindergartners and their birthday months.

Input	Allie	Julie	Danny	Juan	Pete	Bob	Samantha
Output	September	December	December	June	August	December	February

The domain of this function is the list of the children's names. To write it as a set, we enclose the items of the set in curly brackets: {Allie, Julie, Danny, Juan, Pete, Bob, Samantha}.

The range of this function is {September, December, June, August, February}.

10. **a.** Change some thing(s) in this table so it is a function.

b. Give the domain of the function.

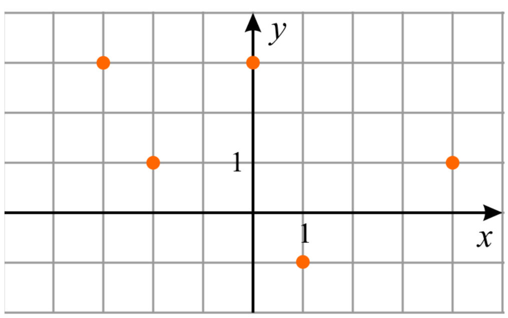
c. Give the range of the function.

Input	Output
Name	**Grade level**
Jenny	8
Pedro	7
Ann	8
Marsha	
Rob	9
Ann	6

11. Let F be the function that maps a number x to $2x + 1$.
Let the set {0, 1, 2, 3, 4, 5} be its domain.
What is its range?

12. Give the domain and range of each function.

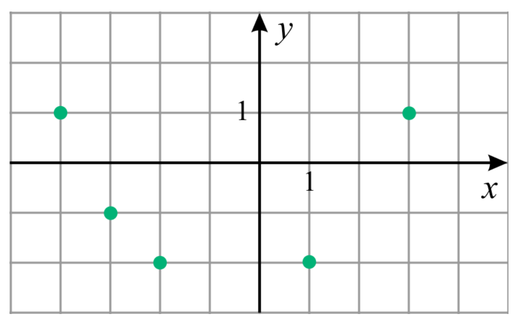
a.



Domain:

Range:

b.



Domain:

Range:

13. Let S be the function that allows any word from this sentence as the input, and the output is the number of letters in it. What is the range of this function?

14. G is a function that maps a number x to $x - 5$.
If the set {0, 5, 10, 15, 20} is its range, what is its domain?

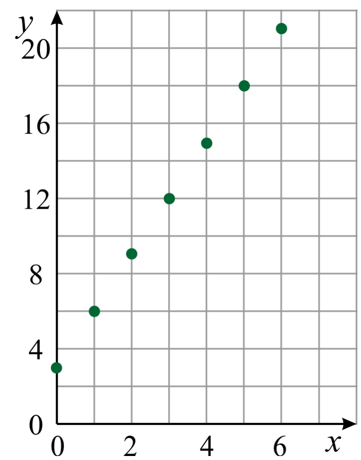
Linear Functions and the Rate of Change 1

If the graph of a function consists of points that fall on a single line, it is a **linear function**.

We will define a linear function in a different manner later, but for now, this is sufficient, so let's look at some examples.

Example 1. The input and output values in the table below define a function. Notice the patterns: the *x*-values increase by ones, and the *y*-values increase by 3s.

Input (x)	0	1	2	3	4	5	6
Output (y)	3	6	9	12	15	18	21



The graph shows that the points fall on a line. This is a linear function.

The **rate of change** of a function is the rate at which the output values change as compared to the change in the input values.

We calculate it as the ratio of $\frac{\text{change in output values}}{\text{change in input values}}$.

In the context of this graph, **rate of change =** $\frac{\text{difference in } y\text{-values}}{\text{difference in } x\text{-values}}$.

In this case, each time the *x*-values increase by 1, the *y*-values increase by 3. **The rate of change is 3/1 = __3__.**

Example 2. The price of bananas is a function of their weight. What is the rate of change?

Weight in kg (input)	0	2	5	10	12	15
Price in \$ (output)	0	5	12.50	25	30	37.50

Check how much the output (price) changes for a certain change in the input (the weight). For example, when the weight increases from 0 to 2 kg, the price increases from \$0 to \$5, or by \$5. This happens also when the weight increases from 10 to 12 kg: the price increases \$5 (from \$25 to \$30).

$$\text{Rate of change} = \frac{\$5}{2\text{ kg}} = \$2.50/\text{kg}$$

Note that if the independent and dependent variables have units, **we include the units in the rate of change**.

This rate of change tells us that for each one-kilogram increase in weight, the price increases by \$2.50.

1. **a.** Calculate the rate of change in example 2, using the increase in weight from 5 to 10 kg, and the corresponding increase in price. Do you get the same rate of change as calculated in the example?

   b. Do the same using the input values 10 kg and 15 kg.

2. What is the rate of change? Don't forget the units!

a.

Input (*t*)	2 hrs	3 hrs	4 hrs	5 hrs	6 hrs	7 hrs
Output (*d*)	$30	$45	$60	$75	$90	$105

b.

Input (*t*)	2 L	4 L	6 L	8 L
Output (*d*)	2.8 kg	5.6 kg	8.4 kg	11.2 kg

3. If a linear function contains the points (4, 15) and (9, 18), what is the rate of change?

4. A train travels at a constant speed, travelling 40 km in 20 minutes. Function D gives the distance (*d*) in kilometres that the train has travelled in *t* hours.

a. Fill in the output values.

***t* (hours)**	0 hrs	1 hr	2 hrs	3 hrs	4 hrs	5 hrs	6 hrs
***d* (km)**							

b. What is the rate of change? Use hours and kilometres.

5. Mr. Stevenson, a gardener, is being paid a base salary of $400 per week for taking basic care of the grounds at a college, plus $25 per hour for certain special tasks. We can model his weekly earnings (E) with the function E = 400 + 25*t* where *t* is the number of hours he works at the special tasks.

a. How much does he get paid if he works five hours at the special tasks in a week?

b. How many hours would he need to work at the special tasks to earn $575 in a week?

c. What is the rate of change of this function?

6. Function D has the rate of change of (7 metres)/(20 minutes), and at 0 minutes, the output value is 0.5 metres.

a. Fill in the table.

Input (t)	0 min	10 min	20 min	30 min	40 min	50 min	60 min
Output (d)	0.5 m						

b. What could this depict?

7. The price of potatoes increases by $10 each time the weight increases by 5 kg.
How do the the rate of change and unit price compare in this situation?

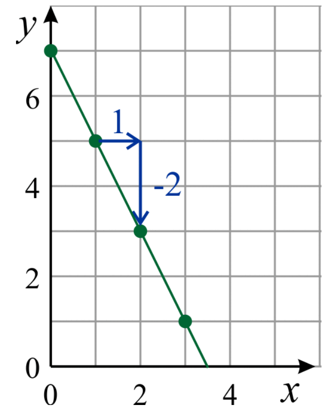
Example 3. The graph shows a plot of a function. To determine the rate of change from a graph, we look at the coordinates of *two* points.

Rate of change = $\dfrac{\text{difference in their } y\text{-values}}{\text{difference in their } x\text{-values}}$

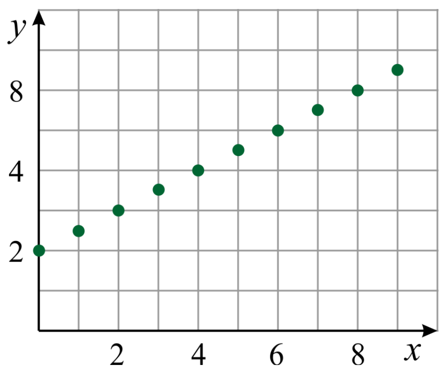
If the function is linear, you can look at *any two* points in the graph in order to determine the rate of change. Here, we use the points (1, 5) and (2, 3).

As the *x*-values **increase by one** (from 1 to 2), the *y*-values ***decrease* by two** (from 5 to 3). This means the difference in the *y*-values is −2.

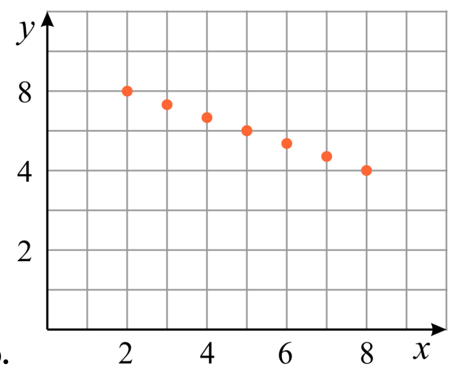
The rate of change is negative, and is $\dfrac{-2}{1}$ = **−2**.



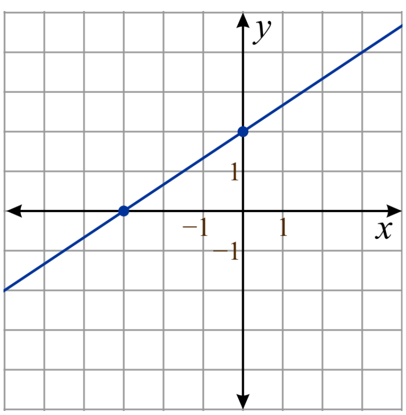
8. Find the rate of change for each function. Note that it can be negative, and/or a fraction.



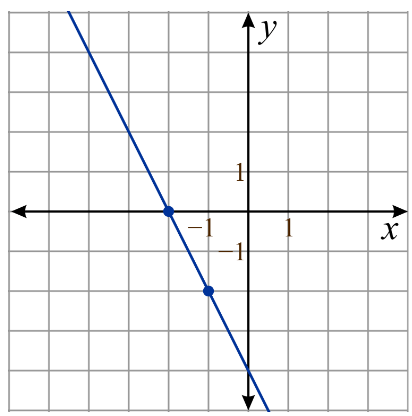
a.



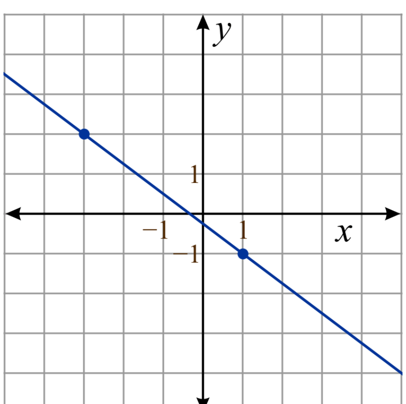
b.



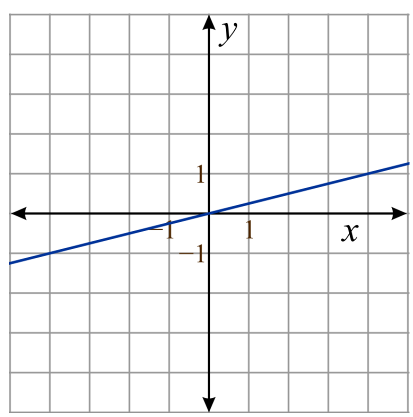
c.



d.



e.



f.